Generalized Kadomtsev-Petviashvili equation with an infinite dimensional symmetry algebra

F. Güngör
Department of Mathematics
Faculty of Science and Letters
Istanbul Technical University
Maslak, Istanbul, Turkey
gungorf@itu.edu.tr

P. Winternitz
Centre de Recherches Mathématiques
Université de Montréal
C. P. 6128, Succ. Centre-ville
Montréal QC H3C 3J7, Canada
wintern@crm.umontreal.ca

Abstract

A generalized Kadomtsev-Petviashvili equation, describing water waves in oceans of varying depth, density and vorticity is discussed. A priori, it involves 9 arbitrary functions of one, or two variables. The conditions are determined under which the equation allows an infinite dimensional symmetry algebra. This algebra can involve up to three arbitrary functions of time. It depends on precisely three such functions if and only if it is completely integrable.

1 Introduction

A generalized Kadomtsev-Petviashvili equation with variable coefficients has been proposed some time ago [1, 2]. The motivation was to describe water waves that propagate in straits, or rivers, rather than on unbounded surfaces, like oceans. This equation was derived from the basic equations of hydrodynamics, namely the Euler equations in three dimensions. The assumptions made were similar to those used when the KP equation is derived for water waves, namely weak nonlinearity, weak dispersion, small amplitudes and propagation in one direction (the x axis) with the waves weakly perturbed in the y direction [3, 4, 5]. However, the derivation allowed for variable depth, the presence of boundaries and of vorticity.

The generalized KP equation thus provides a description of surface waves in a more realistic situation than the KP itself. The additional terms and the variable coefficients make it possible to treat straits of varying width and depth, variable density and to take vorticity into account.

It was shown in the second article [2] that in special cases, corresponding to specific geophysical situations, the generalized KP equation is integrable. It then allows for soliton solutions with "horse-shoe" shaped wave crests. Such waves are indeed observed, for instance issuing from straits opening up into seas or oceans [6, 7].

The purpose of this article is to study the group theoretical properties of a generalized KP equation that is somewhat more general than the one introduced

earlier [1, 2], namely

$$\begin{aligned}
&[u_t + p(t)uu_x + q(t)u_{xxx}]_x + \sigma(y,t)u_{yy} + a(y,t)u_y \\
&+ b(y,t)u_{xy} + c(y,t)u_{xx} + e(y,t)u_x + f(y,t)u + h(y,t) = 0.
\end{aligned} \tag{1.1}$$

We assume that in some neighbourhood we have

$$p(t) \neq 0, \quad q(t) \neq 0, \quad \sigma(y,t) \neq 0. \tag{1.2}$$

The other functions in (1.1) are arbitrary. More specifically, our aim is to determine the cases when eq. (1.1) has an infinite-dimensional symmetry group. The motivation for this is two-fold. On one hand, the existence of an infinite-dimensional symmetry group makes it possible to use Lie group theory to obtain large classes of solutions. On the other hand, integrable equations in 2+1 dimensions typically have Kac-Moody-Virasoro symmetry algebras involving several arbitrary functions of time [8, 9, 10, 11, 12]. This is true for all equations of the KP hierarchy [12]. Moreover, these Kac-Moody-Virasoro symmetries can be directly extracted from a much larger set of symmetries that includes higher symmetries as well as nonlocal ones [12, 13].

Eq. (1.1) with $(p = q = 1, f = h = 0)$ was studied from a different point of view by Clarkson [14] who showed that the equation has the Painlevé property [15, 16, 17] if and only if it can be transformed by a point transformation into the KP equation itself.

In Section 2 we introduce "allowed transformations" that take equations of the form (1.1) into other equations of the same class. That is, they may change the unspecified functions in eq. (1.1), but not introduce other terms, or dependence on other variables. The allowed transformations are used to simplify eq. (1.1) and transform it into eq. (2.5) that we call the "canonical generalized KP equation" (CGKP equation). In Section 3 we determine the general form of the symmetry algebra of the CGKP equation and obtain the determining equations for the symmetries. In Section 4 we establish the most general conditions under which the CGKP equation is invariant under arbitrary reparametrization of time. This means that the symmetry algebra contains the Virasoso algebra as a subalgebra. We show that this Virasoro algebra is present if and only if the CGKP equation can be transformed into the Kadomtsev-Petviashvili equation itself by a point transformation. Section 5 is devoted to the case when the CGKP equation is invariant under a Kac-Moody algebra, (but not under a Kac-Moody-Virasoro one). Some conclusions are presented in Section 6.

2 Allowed transformations and a canonical generalized Kadomtsev-Petviashvili equation

By definition "allowed transformations" are point transformations $(x, y, t, u) \to (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u})$ that take eq. (1.1) into another equation of the same type. That is, the transformed equation will be the same as eq. (1.1), but the arbitrary functions can be different. The typical features of the equation are that the new functions $\tilde{p}(\tilde{t})$ and $\tilde{q}(\tilde{t})$ depend on $\tilde{t}$ alone, the others on $\tilde{y}$ and $\tilde{t}$, but no $\tilde{x}$ dependence is introduced. The only $\tilde{t}$-derivative is $\tilde{u}_{\tilde{x}\tilde{t}}$, the only nonlinear term is $\tilde{p}(\tilde{t})(\tilde{u}\tilde{u}_{\tilde{x}})_{\tilde{x}}$ and the only derivative higher than a second order one is $\tilde{q}(\tilde{t})\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}$. These conditions are very restrictive and they imply that the "allowed transformations" have the form

$$\begin{aligned} &u(x,y,t) = R(t)\tilde{u}(\tilde{x},\tilde{y},\tilde{t}) - \frac{\dot{\alpha}}{\alpha p}x + S(y,t), \\ &\tilde{x} = \alpha(t)x + \beta(y,t), \quad \tilde{y} = Y(y,t), \quad \tilde{t} = T(t), \\ &\alpha \neq 0, \quad R \neq 0 \quad Y_y \neq 0, \quad \dot{T} \neq 0, \quad \dot{\alpha} f(y,t) = 0. \end{aligned} \tag{2.1}$$

The coefficients in the transformed equation satisfy

$$\begin{aligned} \tilde{p}(\tilde{t}) &= p(t)\frac{R\alpha}{\dot{T}}, \quad \tilde{q}(\tilde{t}) = q(t)\frac{\alpha^3}{\dot{T}}, \\ \tilde{\sigma}(y,t) &= \sigma(y,t)\frac{Y_y^2}{\alpha\dot{T}}, \\ \tilde{a}(\tilde{y},\tilde{t}) &= \frac{1}{\alpha\dot{T}}\{aY_y + \sigma Y_{yy}\}, \\ \tilde{b}(\tilde{y},\tilde{t}) &= \frac{1}{\alpha\dot{T}}\{(b\alpha + 2\sigma\beta_y)Y_y + \alpha Y_t\}, \\ \tilde{c}(\tilde{y},\tilde{t}) &= \frac{1}{\alpha\dot{T}}\{c\alpha^2 + \beta_t\alpha + p(t)S\alpha^2 + \sigma\beta_y^2 + b\alpha\beta_y\}, \\ \tilde{e}(\tilde{y},\tilde{t}) &= \frac{1}{\alpha R\dot{T}}\{R\alpha e - R\dot{\alpha} + \dot{R}\alpha + aR\beta_y + \sigma R\beta_{yy}\}, \\ \tilde{f}(\tilde{y},\tilde{t}) &= \frac{1}{\alpha\dot{T}}f, \\ \tilde{h}(\tilde{y},\tilde{t}) &= \frac{1}{\alpha R\dot{T}}\{h - \frac{d}{dt}\left(\frac{\dot{\alpha}}{\alpha p}\right) + p\left(\frac{\dot{\alpha}}{\alpha p}\right)^2 + \sigma S_{yy} + aS_y + fS - e\frac{\dot{\alpha}}{\alpha p}\}. \end{aligned} \tag{2.2}$$

We now choose the functions $R(t), T(t)$ and $Y(y,t)$ in eq. (2.1) to satisfy

$$\begin{aligned} &\dot{T}(t) = q(t)\alpha^3(t), \quad R(t) = \frac{q}{p}\alpha^2, \\ &Y_y = \alpha^2\sqrt{\left|\frac{q(t)}{\sigma(y,t)}\right|} \end{aligned} \tag{2.3}$$

and thus normalize

$$\tilde{p}(\tilde{t}) = 1, \quad \tilde{q}(\tilde{t}) = 1, \quad \tilde{\sigma}(\tilde{y}, \tilde{t}) = \varepsilon = \mp 1. \tag{2.4}$$

By an appropriate choice of the functions $\beta(y, t)$ and $S(y, t)$ we can arrange to have

$$\tilde{e}(\tilde{y}, \tilde{t}) = \tilde{h}(\tilde{y}, \tilde{t}) = 0.$$

Finally, equation (1.1) is reduced to its canonical form

$$\begin{aligned}(u_t + uu_x + u_{xxx})_x + \varepsilon u_{yy} + a(y,t)u_y + b(y,t)u_{xy} \\ + c(y,t)u_{xx} + f(y,t)u = 0, \quad \varepsilon = \pm 1.\end{aligned} \tag{2.5}$$

With no loss of generality we can restrict our study to symmetries of eq. (2.5). All results obtained for eq. (2.5) can be transformed into results for eq. (1.1), using the transformations (2.1). We shall call eq. (2.5) the "canonical generalized KP equation" (CGKP).

Allowed transformations were used earlier in a very similar manner for a variable coefficient Korteweg-de-Vries equation [18, 19].

3 Determining equations for the symmetries

We restrict ourselves to Lie point symmetries. The Lie algebra of the symmetry group is realized by vector fields of the form

$$\hat{\mathbf{V}} = \xi \partial_x + \eta \partial_y + \tau \partial_t + \phi \partial_u, \tag{3.1}$$

where ξ, η, τ and ϕ are functions of x, y, t and u. The algorithm for determining the form of the vector field $\hat{\mathbf{V}}$ for any given system of differential equations goes back to S. Lie and is described in any book on the subject (see e.g. ref. [20]). Many computer packages exist that realize this algorithm. We use the one described in ref. [21]. They provide a partially solved set of determining equations. This is an overdetermined set of linear partial differential equations for the coefficients ξ, η, τ and ϕ in eq. (3.1).

For eq. (2.5) we obtain 17 equations of which 11 do not involve the functions a, b, c and f. These can be solved to yield

$$\begin{aligned}\xi(x,y,t,u) &= \frac{1}{3}\dot{\tau}x + \xi_0(y,t), \\ \eta(x,y,t,u) &= \frac{2}{3}\dot{\tau}y + \eta_0(t), \\ \tau(x,y,t,u) &= \tau(t), \\ \phi(x,y,t,u) &= -\frac{2}{3}\dot{\tau}u + \frac{1}{3}\ddot{\tau}x + S(y,t).\end{aligned} \tag{3.2}$$

One of the remaining determining equations can be used to determine the function $S(y,t)$ to be

$$S(y,t) = -\tau c_t - (\frac{2}{3}\dot{\tau}y + \eta_0)c_y + \xi_{0,t} + b\xi_{0,y} - \frac{2}{3}c\dot{\tau}. \tag{3.3}$$

The remaining determining equations for $\tau(t)$, $\eta(t)$ and $\xi_0(y,t)$ are

$$3\tau a_t + (2\dot{\tau}y + 3\eta_0)a_y + 2a\dot{\tau} = 0, \tag{3.4}$$

$$-3\dot{\eta}_0 - 2y\ddot{\tau} + 3\tau b_t + (2\dot{\tau}y + 3\eta_0)b_y + b\dot{\tau} - 6\varepsilon\xi_{0,y} = 0, \tag{3.5}$$

$$\ddot{\tau} + 3a\xi_{0,y} + 3\varepsilon\xi_{0,yy} = 0, \tag{3.6}$$

$$f\ddot{\tau} = 0, \tag{3.7}$$

$$4f\dot{\tau} + 3f_t\tau + f_y(2\dot{\tau}y + 3\eta_0) = 0, \tag{3.8}$$

$$\dddot{\tau} + 3fS + 3aS_y + 3\varepsilon S_{yy} = 0, \tag{3.9}$$

where $S(y,t)$ of eq. (3.3) should be substituted into eq. (3.9).

It is beyond the scope of this article to perform a complete analysis of eqs. (3.3),...,(3.9) for arbitrary (given) functions a, b, c and f. Rather, we shall determine the conditions on these functions that permit the symmetry algebra to be infinite-dimensional. This will happen when at least one of the functions $\tau(t)$, $\eta_0(t)$ and $\xi_0(y,t)$ remains an arbitrary function of at least one variable.

From eq. (3.7) we see immediately that τ can be arbitrary only if $f(y,t)$ satisfies $f(y,t) = 0$. From eq. (3.6) we see $\xi_0(y,t)$ may be an arbitrary function of t, but never of y (we have $\varepsilon = \pm 1$).

4 Virasoro symmetries of the CGKP equation

The canonical generalized KP equation (2.5) will be invariant under a transformation group, the Lie algebra of which is isomorphic to a Virasoro algebra if the function τ in (3.2) remains free. Thus, we are looking for conditions on the coefficients a, b, c and f that allow equations (3.4),...(3.9) to be solved without imposing any conditions on $\tau(t)$.

From eq. (3.7) we see that τ is linear in t, unless we have $f(y,t) \equiv 0$. Once this condition is imposed, equations (3.7) and (3.8) are solved identically. Eq. (3.4) leaves $\tau(t)$ free if either we have $a = 0$, or $a = a_0(y + \lambda(t))^{-1}$ where $a_0 \neq 0$ is a constant and $\lambda(t)$ is some function of t. We investigate the two

cases separately. First let us assume

$$a = \frac{a_0}{y + \lambda(t)}, \quad a_0 \neq 0. \tag{4.1}$$

Then we view eq. (3.4) as an equation for $\eta_0(t)$ and obtain

$$\eta_0(t) = \frac{1}{3}(2\lambda\dot{\tau} - 3\dot{\lambda}\tau). \tag{4.2}$$

Eq. (3.6) allows us to determine $\xi_0(y,t)$ in terms of $\tau(t)$. Three possibilities occur:
1.) $a_0\varepsilon \neq \pm 1$

$$\xi_0 = -\frac{\varepsilon\ddot{\tau}(y+\lambda)^2}{6(1+a_0\varepsilon)} + \frac{\mu_1(t)(y+\lambda)^{-a_0\varepsilon+1}}{1-a_0\varepsilon} + \mu_0(t). \tag{4.3}$$

2.) $a_0\varepsilon = 1$

$$\xi_0 = -\frac{\varepsilon\ddot{\tau}}{12}(y+\lambda)^2 + \mu_1(t)\ln(y+\lambda) + \mu_0(t). \tag{4.4}$$

3.) $a_0\varepsilon = -1$

$$\xi_0 = -\frac{\varepsilon\ddot{\tau}}{12}(y+\lambda)^2[2\ln(y+\lambda) - 1] + \mu_1(t)(y+\lambda)^2 + \mu_0(t). \tag{4.5}$$

We must now put ξ_0 of (4.3), (4.4) or (4.5) into eq. (3.5) and solve the obtained equation for $\mu_1(t)$. The expression for $\mu_1(t)$ must be independent of y for all values of τ. Moreover, for $\tau(t)$ to remain free, there must be no relation between $b(y,t)$ and $\tau(t)$. These conditions cannot be satisfied for any value of $a_0\varepsilon$. Hence, if $a(y,t)$ is as in eq. (4.1) the generalized KP equation (2.5) does not allow a Virasoro algebra.

The other case to consider is $a = 0$ (in addition to $f = 0$). Eq. (3.6) is easily solved in this case and we obtain

$$\xi_0(y,t) = -\frac{\varepsilon\ddot{\tau}}{6}y^2 + \mu_1(t)y + \mu_0(t) \tag{4.6}$$

with $\mu_1(t)$ and $\mu_0(t)$ arbitrary. We insert $\xi_0(y,t)$ into eq. (3.5) and solve for $\mu_1(t)$. This is possible if and only if we have $b = b(t)$ (no dependence on y). We obtain

$$\mu_1(t) = \frac{\varepsilon}{6}(b\tau + 3\dot{b}\tau - 3\dot{\eta}_0). \tag{4.7}$$

We put the expression (3.3) for $S(y,t)$ into eq. (3.9) and obtain

$$-2\dot{\tau}(yc_{yyy} + 3c_{yy}) - 3c_{tyy}\tau + 3\eta_0 c_{yyy} = 0. \tag{4.8}$$

Eq. (4.8) restricts the form of $\tau(t)$, unless we have

$$c(y,t) = c_0(t) + c_1(t)y. \tag{4.9}$$

The result is that for $a = f = 0$, $b = b(t)$, $c = c_0(t) + c_1(t)y$ all equations (3.4),...,(3.9) are solved with $\tau(t)$, $\eta_0(t)$ and $\mu_0(t)$ arbitrary. Let us sum up the result as two theorems.

Theorem 1 *The canonical generalized KP equation* (2.5) *allows the Virasoro algebra as a symmetry algebra if and only if the coefficients satisfy*

$$a = f = 0, \quad b = b(t), \quad c = c_0(t) + c_1(t)y. \tag{4.10}$$

Theorem 2 *The canonical generalized KP equation*

$$(u_t + uu_x + u_{xxx})_x + \varepsilon u_{yy} + b(t)u_{xy} + [c_0(t) + c_1(t)y]u_{xx} = 0 \tag{4.11}$$

with $\varepsilon = \pm 1$, and $b(t)$, $c_0(t)$ and $c_1(t)$ arbitrary smooth functions is invariant under an infinite-dimensional Lie point symmetry group. Its Lie algebra has a Kac-Moody-Virasoro structure. It is realized by vector fields of the form

$$\hat{\mathbf{V}} = T(\tau) + X(\xi) + Y(\eta), \tag{4.12}$$

where $\tau(t)$, $\xi(t)$ and $\eta(t)$ are arbitrary smooth functions of time and we have

$$\begin{aligned} T(\tau) &= \tau(t)\partial_t + \frac{1}{6}[3\varepsilon\dot{b}y\tau + (2x + \varepsilon by)\dot{\tau} - \varepsilon\ddot{\tau}y^2]\partial_x \\ &+ \frac{2}{3}\dot{\tau}\partial_y + \frac{1}{6}\{[-6\dot{c}_0 + 3\varepsilon b\dot{b} + (-6\dot{c}_1 + 3\varepsilon\ddot{b})y]\tau \\ &+ [-4u + \varepsilon b^2 - 4c_0 + 4(\varepsilon\dot{b} - 2c_1)y]\dot{\tau} + (2x - \varepsilon by)\ddot{\tau} \\ &- \varepsilon y^2 \dddot{\tau}\}\partial_u, \end{aligned} \tag{4.13}$$

$$X(\xi) = \xi(t)\partial_x + \dot{\xi}(t)\partial_u, \tag{4.14}$$

$$Y(\eta) = \eta(t)\partial_y - \frac{\varepsilon}{2}\dot{\eta}(t)y\partial_x - \frac{1}{2}[2c_1\eta + \varepsilon b\dot{\eta} + \varepsilon y\ddot{\eta}]\partial_u. \tag{4.15}$$

The form of eq. (4.11) and its symmetry algebra (4.12),...,(4.15) suggests that it might be transformable into the KP equation itself. This is indeed the case. The transformation

$$\begin{aligned} u(x, y, t) &= \tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) + (\frac{\varepsilon}{2}\dot{b} - c_1)y - c_0 + \frac{\varepsilon}{4}b^2, \\ \tilde{x} &= x - \frac{\varepsilon b}{2}y, \quad \tilde{y} = y, \quad \tilde{t} = t \end{aligned} \tag{4.16}$$

takes eq. (4.11) into the KP equation itself, i.e. into eq. (4.11) with $b = c_0 = c_1 = 0$. The transformation (4.16) also transforms the Lie algebra (4.12),...,(4.15) into the symmetry algebra [8] of the KP equation.

We have obtained the following result.

Theorem 3 *The GKP equation* (1.1) *is invariant under a Lie point symmetry group, the Lie algebra of which contains a Virasoro algebra as a subalgebra, if and only if it can be transformed into the KP equation itself by a point transformation.*

5 Kac-Moody symmetries of the CGKP equation

In section 4 we have shown that if the symmetry algebra of the canonical generalized Kadomtsev-Petviashvili contains a Virasoro algebra, then it also contains a Kac-Moody algebra. In this section we will determine the conditions on the functions $a(y,t)$, $b(y,t)$, $c(y,t)$ and $f(y,t)$ under which the CGKP equation allows a Kac-Moody algebra, without allowing a Virasoro one. Thus, the function $\tau(t)$ will not be free, but $\eta_0(t)$ of eq. (3.2) will be free, or $\xi_0(y,t)$ will involve at least one free function of t.

5.1 The function $\eta_0(t)$ free

Eq. (3.4) will relate η_0 and $a(y,t)$ unless we have $a_y = 0$. Hence we put $a_y = 0$. For $a = a(t) \neq 0$ eq. (3.4) implies $\tau(t) = \tau_0 a^{-3/2}$. Eq. (3.6) then yields

$$\xi = \xi_1(t) e^{-a\varepsilon y} - \frac{\ddot{\tau}}{3a} y + \xi_0(t).$$

Eq. (3.5) then provides a relation between $\eta_0(t)$ and $b(y,t)$. Hence $\eta_0(t)$ is not free. Thus, if $\eta_0(t)$ is to be a free function, we must have $a(y,t) = 0$. Eq. (3.4) is satisfied identically. From eq. (3.6) we have

$$\xi_0(y,t) = -\frac{\varepsilon}{6}\ddot{\tau} y^2 + \rho(t) y + \sigma(t). \tag{5.1}$$

Eq. (3.5) will leave η_0 free only if we have

$$b(y,t) = b_1(t) y + b_0(t), \tag{5.2}$$

$$\rho(t) = \frac{\varepsilon}{6}(-3\dot{\eta}_0 + 3\tau \dot{b}_0 + 3\eta_0 b_1 + b_0 \dot{\tau}), \tag{5.3}$$

$$(\tau b_1)^{\cdot} = 0. \tag{5.4}$$

For $f \neq 0$ we have $\ddot{\tau} = 0$ and eq (3.9) will relate $\eta(t)$ to $c(y,t)$, b_1 and b_0. Thus, for $\eta_0(t)$ to be free, we must have $f(y,t) = 0$. Eq. (3.9) reduces to

$$\dddot{\tau} + 3\varepsilon S_{yy} = 0.$$

This equation is only consistent if we have

$$c(y,t) = c_2(t) y^2 + c_1(t) y + c_0(t), \tag{5.5}$$

$$\varepsilon b_1 \ddot{\tau} + 3\dot{c}_2 \tau + 6\dot{\tau} c_2 = 0. \tag{5.6}$$

The only equation that remains to be solved is eq. (5.6). Both functions $\eta_0(t)$ and $\sigma(t)$ remain free. If we have $b_1 = 0$, $c_2 = 0$, then the function $\tau(t)$ is also free and we reobtain the entire Kac-Moody-Virasoro algebra of Section 4. The most general CGKP equation allowing $\eta_0(t)$ to be a free function is obtained if eq. (5.6) is solved identically by putting $\tau = 0$. Then $\eta_0(t)$ and $\sigma(t)$ are arbitrary. Using eq. (3.2) and the above results we obtain the following theorem.

Theorem 4 *The equation*

$$\begin{aligned}&(u_t + uu_x + u_{xxx})_x + \varepsilon u_{yy} + (b_1(t)y + b_0(t))u_{xy} \\ &\quad + (c_2(t)y^2 + c_1(t)y + c_0(t))u_{xx} = 0,\end{aligned} \tag{5.7}$$

where $\varepsilon = \pm 1$ and b_0, b_1, c_0, c_1, c_2 are arbitrary functions of t, is the most general canonical generalized KP equation, invariant under an infinite-dimensional Lie point symmetry group depending on two arbitrary functions. Its Lie algebra has a Kac-Moody structure [22] and is realized by vector fields of the form

$$\hat{\mathbf{V}} = X(\xi) + Y(\eta), \tag{5.8}$$

where $\xi(t)$ and $\eta(t)$ are arbitrary smooth functions of time and

$$X(\xi) = \xi\partial_x + \dot{\xi}\partial_u, \tag{5.9}$$

$$\begin{aligned}Y(\eta) = {}& \eta\partial_y + \frac{\varepsilon}{2}y(-\dot{\eta} + b_1\eta)\partial_x + \{[-2c_2\eta \\ &+ \frac{\varepsilon}{2}(-\ddot{\eta} + \dot{b}_1\eta + b_1^2\eta)]y - c_1\eta + \frac{\varepsilon}{2}b_0(-\dot{\eta} + b_1\eta)\}\partial_u.\end{aligned} \tag{5.10}$$

Several comments are in order:

1. The symmetry algebra of eq. (5.7) is larger for special cases of the functions b_1, b_0, c_2, c_1 and c_0. Thus, if $c_2 = b_1 = 0$ we recover the entire Kac-Moody-Virasoro algebra of Theorem 2 since eq. (5.6) is satisfied identically (i.e. the function $\tau(t)$ is also free). The two other special cases are $b_1 = 0$, $c_2 \neq 0$ and $b_1 \neq 0$ (see below).

2. Eq. (5.7) can be further simplified by allowed transformations. Indeed, let us restrict the transformation (2.1) to

$$\begin{aligned}&u(x, y, t) = \tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) + S_1(t)y + S_0(t), \\ &\tilde{x} = x + \beta_1(t)y + \beta_0(t), \quad \tilde{y} = y + \gamma(t), \quad \tilde{t} = t.\end{aligned} \tag{5.11}$$

For any functions $b_1(t)$ and $c_2(t)$ we can choose $S_1, S_0, \beta_0, \beta_1$ and γ to set b_0, c_1 and c_0 equal to zero. Thus, with no loss of generality, we can set

$$b_0(t) = c_1(t) = c_0(t) = 0 \tag{5.12}$$

in eq. (5.7), (5.9) and (5.10).

3. Let us now consider the cases when eq. (5.7) has an additional symmetry.

Case 1. $b_1 = 0, c_2 \neq 0$

We assume (5.12) is already satisfied. From eq. (5.6) we have $\tau = \tau_0 c_2^{-1/2}$. The additional symmetry is

$$\begin{aligned} T = & \frac{1}{\sqrt{c_2}}\partial_t + [\frac{1}{3}\frac{d}{dt}\left(\frac{1}{\sqrt{c_2}}\right)x - \frac{\varepsilon}{6}\frac{d^2}{dt^2}\left(\frac{1}{\sqrt{c_2}}\right)y^2]\partial_x \\ & + \frac{2}{3}\frac{d}{dt}\left(\frac{1}{\sqrt{c_2}}\right)y\partial_y + \{-\frac{2}{3}\frac{d}{dt}\left(\frac{1}{\sqrt{c_2}}\right)u + \frac{1}{3}\frac{d^2}{dt^2}\left(\frac{1}{\sqrt{c_2}}\right)x \\ & - \frac{\varepsilon}{6}\frac{d^3}{dt^3}\left(\frac{1}{\sqrt{c_2}}\right)y^2\}\partial_u, \end{aligned} \tag{5.13}$$

Case 2. $b_1 \neq 0$, $b_0 = c_1 = c_0 = 0$

Eq. (5.4) implies $\tau = \tau_0 b^{-1}$ and eq. (5.6) provides the constraint

$$c_2 = \frac{1}{3}(\varepsilon \dot{b}_1 + k b_1^2) \tag{5.14}$$

where k is a constant. The additional element of the symmetry algebra in this case is

$$\begin{aligned} T = & \frac{1}{b_1}\partial_t + [\frac{1}{3}\frac{d}{dt}\left(\frac{1}{b_1}\right)x - \frac{\varepsilon}{6}\frac{d^2}{dt^2}\left(\frac{1}{b_1}\right)y^2]\partial_x + \frac{2}{3}\frac{d}{dt}\left(\frac{1}{b_1}\right)y\partial_y \\ & + [-\frac{2}{3}\frac{d}{dt}\left(\frac{1}{b_1}\right)u + \frac{1}{3}\frac{d^2}{dt^2}\left(\frac{1}{b_1}\right)x - \frac{\varepsilon}{6}\frac{d^3}{dt^3}\left(\frac{1}{b_1}\right)y^2]\partial_u. \end{aligned} \tag{5.15}$$

5.2 One free function in symmetry algebra

We have established that if $\tau(t)$ is free in eq. (3.2), then there are three free functions. If τ is not free, but $\eta_0(t)$ is, then there are two free functions. Now let $\tau(t)$ and $\eta_0(t)$ be constrained by the determining equations, but let some freedom remain in the function $\xi_0(y,t)$.

First of all we note that if we put

$$\tau = 0, \quad \eta_0 = 0, \quad \xi_0(y,t) = \xi(t) \tag{5.16}$$

in eq. (3.2) then eqs. (3.4),...,(3.8) are satisfied identically and eq. (3.9) reduces to

$$f\dot{\xi} = 0. \tag{5.17}$$

Hence

$$X(\xi) = \xi(t)\partial_x + \dot{\xi}(t)\partial_u, \tag{5.18}$$

with $\xi(t)$ arbitrary, generates Lie point symmetries of the CGKP equation for $f(y,t)=0$ and any functions $a(y,t), b(y,t)$, and $c(y,t)$.

For $f \neq 0$ we have $\tau = \tau_1 t + \tau_0$ from eq. (3.7). Eq. (3.6) then determines the y dependence of ξ_0. Two possibilities occur:

(i)

$$a = 0, \quad \xi_0 = \rho(t)y + \sigma(t) \tag{5.19}$$

(ii)

$$a \neq 0, \quad a(y,t) \equiv -\varepsilon \frac{A_{yy}}{A_y}, \quad A_{yy} \neq 0, \quad \xi_0 = \sigma_1(t)A(y,t) + \sigma_2(t). \tag{5.20}$$

We skip the details here and just state that the remaining equations (3.5), (3.8) and (3.9) do not allow any solutions with free functions.

We state this result as a theorem.

Theorem 5 *The CGKP equation* (2.5) *is invariant under an infinite-dimensional Abelian group generated by the vector field* (5.18) *for* $f(y,t) = 0$ *and* a, b, c *arbitrary.*

Theorems 2, 4 and 5 sum up all cases when the symmetry algebra of the CGKP equation is infinite-dimensional.

6 Applications and conclusions

We have identified all cases when the generalized KP equation has an infinite-dimensional symmetry group. Let us now discuss the implications of this result.

6.1 Equation with Kac-Moody-Virasoro symmetry algebra

We have shown that eq. (4.11) is the most general CGKP equation invariant under a Kac-Moody-Virasoro group. Moreover, it can be transformed into the KP equation itself. It follows that any solution of the KP equation can be transformed into a solution of eq. (4.11). The corresponding transformation will however not take solitons into solitons. More generally, it will not preserve boundary conditions. The Lax pair for the KP equation is of course well known and much studied [5]. The transformation inverse to (4.16) will take it into a Lax pair for eq. (4.11). The transformed Lax pair can then be simplified by a redefinition of the wave function figuring in it. In turn, this Lax pair can be used to obtain new solutions of eq. (4.11). Work on this problem is in progress

but goes beyond the scope of the present article. We note here that eq. (4.11) can also be transformed into the integrable cylindrical KP equation

$$(u_t + uu_x + u_{xxx})_x + \frac{1}{2t}u_x + \frac{\varepsilon}{4t^2}u_{yy} = 0, \tag{6.1}$$

or one of its generalizations, which have been studied extensively [23, 9].

6.2 Equation with nonabelian Kac-Moody symmetry algebra

The symmetry algebra (5.8) of eq. (5.7) is infinite-dimensional and nonabelian. Indeed, we have

$$[Y(\eta_1), Y(\eta_2)] = X(\xi), \quad \xi = -\frac{\varepsilon}{2}(\eta_1\dot{\eta}_2 - \dot{\eta}_1\eta_2). \tag{6.2}$$

Unless we have $b_1 = c_2 = 0$ equation (5.7) is not integrable. We can however apply the method of symmetry reduction to obtain particular solutions. The operator $X(\xi)$ of eq. (5.9) generates the transformations

$$\tilde{x} = x + \lambda\xi(t), \quad \tilde{y} = y, \quad \tilde{t} = t, \quad \tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) = u(x, y, t) + \lambda\dot{\xi}(t), \tag{6.3}$$

where λ is a group parameter. We see that (6.3) is a transformation to a frame moving with an arbitrary acceleration in the x direction. For ξ constant this is a translation, for ξ linear in t this is a Galilei transformation. An invariant solution will have the form

$$u = \frac{\dot{\xi}}{\xi}x + F(y, t). \tag{6.4}$$

Substituting into eq. (5.7) we obtain the family of solutions

$$u = \frac{\dot{\xi}}{\xi}x - \frac{\varepsilon}{2}\frac{\ddot{\xi}}{\xi}y^2 + \rho(t)y + \sigma(t) \tag{6.5}$$

with $\rho(t)$ and $\sigma(t)$ arbitrary.

The transformation corresponding to the general element $Y(\eta) + X(\xi)$ with $\eta \neq 0$ is easy to obtain, but more difficult to interpret. An invariant solution will have the form

$$\begin{aligned} u &= [-c + \frac{\varepsilon}{4}(\dot{b} + b^2 - \frac{\ddot{\eta}}{\eta})]y^2 + \frac{\dot{\xi}}{\eta}y + F(z, t) \\ z &= x + \frac{\varepsilon}{4}(-b + \frac{\dot{\eta}}{\eta})y^2 - \frac{\xi}{\eta}y. \end{aligned} \tag{6.6}$$

We have put $b_1 = b,\ c_2 = c,\ b_0 = c_1 = c_0 = 0$, which can be done with no loss of generality. We now put u of eq. (6.6) into eq. (5.7) (for $c_1 = c_0 = b_0 = 0$) and obtain the reduced equation

$$(F_t + FF_z + F_{zzz})_z + \varepsilon\frac{\xi^2}{\eta^2}F_{zz} + \frac{1}{2}(\frac{\dot\eta}{\eta} - b)F_z - 2c\varepsilon + \frac{1}{2}(\dot b + b^2 - \frac{\ddot\eta}{\eta}) = 0. \quad (6.7)$$

In general, eq. (6.7) is not integrable. Putting

$$\begin{aligned} F(z,t) &= \tilde F(\tilde z,\tilde t), \quad \tilde z = z + \beta(t), \quad \tilde t = t, \\ \dot\beta(t) &= -\varepsilon\frac{\xi^2}{\eta^2} \end{aligned} \quad (6.8)$$

we eliminate the F_{zz} term. Choosing $\dot\eta/\eta = b(t)$ we obtain the equation

$$(F_t + FF_z + F_{zzz})_z = 2\varepsilon c(t), \quad (6.9)$$

an equation that is not integrable (for $c \neq 0$) but that has been extensively studied [3, 24]. We note that for $c = 0$ (6.9) reduces to the KdV equation, even though the reduced equation (5.7) is not integrable for $b_1(t) \equiv b(t) \neq 0$.

6.3 Equation with Abelian Kac-Moody symmetry algebra

Let us now consider the CGKP equation (2.5) with $f = 0$. It is invariant under the transformations generated by $X(\xi)$ of eq. (5.18). The invariant solution has the form (6.4) and $F(t,y)$ will satisfy the linear equation

$$\varepsilon F_{yy} + a(y,t)F_y + \frac{\dddot\xi}{\xi} = 0. \quad (6.10)$$

Eq. (6.10) can be solved for any function $a(y,t)$ and the general solution is

$$\begin{aligned} F(y,t) = \alpha(t)A(y,t) + \beta(t) + \varepsilon\frac{\dddot\xi}{\xi}\Big[\int_{y_0}^{y}\frac{A(y',t)}{A_{y'}(y',t)}\,dy' \\ - \Big(\int_{y_0}^{y}\frac{1}{A_{y'}(y',t)}\,dy'\Big)A(y,t)\Big], \end{aligned} \quad (6.11)$$

where $\alpha(t)$, $\beta(t)$ (and also $\xi(t)$) are arbitrary functions of t. We have put

$$A(y,t) = \int_{y_0}^{y}[\exp\big(-\varepsilon\int_{y_0}^{y'}a(y'',t)dy''\big)]dy', \quad (6.12)$$

i.e. $A(y,t)$ is a particular solution of (6.10) for $\dddot\xi = 0$ (the homogeneous equation).

For $a(y,t)=0$ the solution (6.11) reduces to (6.5), as it should.

One situation in which solutions of the type (6.4) could be relevant is that of water propagating parallel to a beach. The origin $y=0$ of the y axis would be somewhere in the deep ocean and $y=y_{\max}$ would be at the shore. The function $a(y,t)$ could be very large for y small, then decrease towards the shore. An example of such a function and the corresponding solution are

$$a(y,t)=-\varepsilon\frac{m(t)}{y},\quad F(y,t)=A(t)y^{m+1}+\frac{\dot{\xi}}{2\varepsilon\xi(m-1)}y^2+C(t),\tag{6.13}$$

where $m(t)$, $A(t)$ and $C(t)$ are arbitrary functions.

6.4 Comments and Outlook

The most ubiquitous symmetry of the generalized KP equation is the transformation (6.3) to an arbitrary frame moving in the x direction. Its presence only requires the coefficient $f(y,t)$ in eq. (1.1) (or (2.5)) to be $f(y,t)\equiv 0$. Invariance of a solution under such a general transformation is very restrictive and leads to solutions that are at most linear in the variable x and have a prescribed y dependence (see solutions (6.5), (6.11), (6.13)). Such solutions may actually physically be meaningful under special conditions. The wave crests $u(x,y,t)=$ const. for fixed t have a parabolic shape in the case of eq. (6.5). Such "horse-shoe" shaped waves coming out of straits are observed. Since $u(x,y,t)$ grows linearly with x, such solutions can only be physical for a finite range of values of x. After that, presumably the conditions under which the GKP was derived no longer hold and the solution breaks, or acquires a different form.

The transformations generated by $Y(\eta)$ leave a more restricted class of GKP equations invariant, those of eq. (5.7). The invariant solutions have the form (6.6) and they are much more general and realistic than the solutions (6.11). They are also much harder to pin down, since it is still necessary to solve eq. (6.7) or (6.9). For general $c(t)$ this is difficult, but for $c(t)=0$ this is just the KdV equation, for arbitrary $b(t)$, as long as we choose $\dot{\eta}/\eta=b(t)$. Any solution of the KdV equation, in particular soliton, or multisoliton solutions will, via eq. (6.6), provide y dependent solutions of the corresponding GKP equation.

According to our opinion, the most important and interesting remaining question is: What can we do with the integrable CGKP (4.11)? This merits a separate investigation, using the tools of soliton theory: the inverse spectral transform and Bäcklund transformations. Both are known for the KP equation [25, 26, 5]. We plan to adapt them to eq. (4.11) and to use them to obtain multisoliton and other physically important solution of this equation. Simple solutions like (6.5) can then serve as input solutions into Bäcklund transformations.

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